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On Necessary Conditions for Optimality in a Banach Space*

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1. INTRODUCTION

Many control problems have a natural Banach space setting. Attempts to generalize the Pontryagin Maximum Principle to this setting have, understandably, met with only partial success. In a rather cryptic paper [3], Egorov, using the method of Rozonoer [9], obtained both local and global "Maximum Principles" for operator equations in Banach spaces. Some of the conditions assumed by Egorov are rather restrictive. For example, in some places he requires the nondifferentiability of certain functions.

In [1], Cesari has extended the work of Egorov to partial differential equations in "Dieudonne–Rashevsky" form.

In this paper, we obtain an analog of the maximum principle for operator equations in a Banach space, allowing more general equations and functionals than does Egorov, and the conjugate equation arises in a very natural way. On the other hand, our method draws heavily on the Fréchet differential calculus, and our differentiability requirements are rather strong. To treat differentiability at boundary points, we incorporate some results in Hestenes [5] on differentiability with respect to a cone, which go over immediately to the infinite dimensional case.

We remark that there are many other works treating necessary conditions in Banach spaces, but varying in one way or another from our approach here. As might be expected, the more concrete the problem considered, the sharper the results obtained. In addition to those already mentioned, we refer to Lions [6], Yu. Egorov [4] and Raitums [7].

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After obtaining our "Maximum Principle," we consider two examples. In the first, a finite dimensional example, our result leads to a precise determination of the optimal control. In the second, a Dirichlet problem, our result easily leads to information on the bang-bang nature of the process.

2. Preliminary Considerations

We assume the reader is familiar with the elementary Fréchet differential calculus on a Banach space, as given, for example, in Dieudonne [2]. We recall some of the basic properties and notation. Let E, F be two Banach spaces, and W open in E. Let $f: W \to F$ be a continuous map. We say f is differentiable at $x_0 \in W$ if there exists a continuous linear map $\Lambda : E \to F$ such that if we write

$$f(x_0 + y) = f(x_0) + \Lambda y + \omega(y), \tag{2.1}$$

then

$$\lim_{\|y\| \to 0} \frac{\| \omega(y) \|}{\| y \|} = 0. \tag{2.2}$$

The (uniquely defined) linear operator Λ is usually written $Df(x_0)$ or $Df(x_0; \cdot)$, and its value at the point y is written as $Df(x_0) \cdot y$ or $Df(x_0; y)$. Let E_i $(i = 1, 2)$ be Banach spaces, and define $E = E_1 \times E_2$. Let W be open in E. Let $f: E \to F$ be continuous. We say f is continuously differentiable in W if Df is continuous in W. Let $(a_1, a_2) \in W$. Then in the usual way we consider the partial mappings $x_1 \to f(x_1, a_2)$ and $x_2 \to f(a_1, x_2)$, and we define the partial derivatives as the derivatives of the partial mappings. We denote the partial derivative of the map $x_1 \to f(x_1, a_2)$ at (a_1, a_2) as $D_1 f(a_1, a_2; \cdot)$ and similarly for $x_2 \to f(a_1, x_2)$.

We shall need the following theorem proved in Dieudonne [2, p. 167]:

Theorem. *Let f be a continuous map of an open subset W of $E_1 \times E_2$ into F. In order that f be continuously differentiable in W, it is necessary and sufficient that f be differentiable at each point with respect to the first and second variables, and that the maps $(x_1, x_2) \to D_1 f(x_1, x_2; \cdot)$ and $(x_1, x_2) \to D_2 f(x_1, x_2; \cdot)$ (of W into $\mathscr{L}(E_1, F)$ and $\mathscr{L}(E_2, F)$) be continuous in W. Then at each $(x_1, x_2) \in W$, $Df(x_1, x_2)$ evaluated at (t_1, t_2) is given by*

$$Df(x_1, x_2; t_1, t_2) = D_1 f(x_1, x_2) \cdot t_1 + D_2 f(x_1, x_2) \cdot t_2 . \tag{2.3}$$

In addition to the validity of the usual chain rule, we shall find the following fact useful. If $T \in \mathscr{L}(B_1, B_2)$ and f is differentiable from $B_1 \to B_1$, then $D(Tf) = T(Df)$.

DEFINITION. Let U be an arbitrary set in E and let $x_0 \in U$. We say ν is an *accessible direction for* x_0 *in* U if and only if

(i) $\| \nu \| = 1$,

(ii) There exists an infinite sequence $\{x_q\}$, $x_q \in U$ with

$$\lim_{q \to \infty} \| x_q - x_0 \| = 0 \qquad \text{and} \qquad \lim_{q \to \infty} \left\| \frac{x_q - x_0}{\| x_q - x_0 \|} - \nu \right\| = 0.$$

Let

$$\begin{aligned} \mathscr{T}_U(x_0) &= \{\nu \mid \nu \text{ is an accessible direction, if any such exist}\} \\ &= \{0\}, \qquad \text{the zero vector, otherwise.} \end{aligned}$$

Let

$$\mathscr{C}_U(x_0) = \{y \mid y = x_0 + \alpha\nu \text{ with } \nu \in \mathscr{T}_U(x_0), \alpha \geqslant 0\}.$$

$\mathscr{C}_U(x_0)$ is called the *tangent cone* of x_0 in U. For example, if $x_0 \in \operatorname{int} U$, then the tangent cone is all of E. The following lemma is essentially given in Hestenes [5] for the finite dimensional case.

LEMMA 1. *Let W be an open set in the Banach space E, and f a continuously differentiable real valued function on W. Then for any $U \subseteq W$, if f has a local minimum at $x_0 \in U$, then*

$$Df(x_0; h) \geqslant 0 \tag{2.4}$$

for all h such that $x_0 + h \in \mathscr{C}_U(x_0)$, with equality if both $x_0 + h$ and $x_0 - h \in \mathscr{C}_U(x_0)$.

Proof. Since $Df(x_0; h)$ is linear in h, it suffices to consider only $h = \nu \in \mathscr{T}_U(x_0)$. Let $\{x_q\}$ be an infinite sequence of points in U converging to x_0 in the direction ν. If x_0 gives a minimum, then $f(x_q) \geqslant f(x_0)$ for q sufficiently large, say $q \geqslant N$. Further, since $x_0 \in U \subset W$, which is open, we conclude from the definition of the derivative, that

$$f(x_q) - f(x_0) = Df(x_0; x_q - x_0) + \omega(x_0, x_q - x_0).$$

Therefore,

$$0 \leqslant \frac{f(x_q) - f(x_0)}{\| x_q - x_0 \|} = Df\left(x_0; \frac{x_q - x_0}{\| x_q - x_0 \|}\right) + \frac{\omega(x_0, x_q - x_0)}{\| x_q - x_0 \|} \tag{2.5}$$

for all q sufficiently large. Since

$$\frac{x_q - x_0}{\| x_q - x_0 \|} \to \nu$$

and the last term in (2.5) tends to zero, passing to the limit in (2.5) yields (2.4). Further, if both ν and $-\nu \in \mathscr{T}_U(x_0)$, then we obtain $Df(x_0; \nu) = 0$. ∎

3. Principal Results

Let B_1, B_2, and B_3 be Banach spaces. Let A be a densely defined linear (not necessarily bounded) operator mapping B_1 onto B_2, and let $f: B_1 \times U \to B_2$ where $U \subset B_3$. Let F be a real valued functional on $B_1 \times U$.

We consider the problem: Given that

$$Ax = f(x, u) \tag{3.1}$$

has a unique solution $x(u)$ for each $u \in U$, we wish to determine necessary conditions that $\tilde{u} \in U$ be an optimal control in the sense that

$$F(x(\tilde{u}), \tilde{u}) \qquad \text{is minimized over } U. \tag{3.2}$$

We shall need the following theorem for the existence and representation of the derivative of the map $u \to x(u)$.

THEOREM 1. *Let B_1, B_2, B_3 and A be as above, and let $f: B_1 \times W \to B_2$, W open in B_3. If*

(i) *$Ax = f(x, u)$ has a unique solution $x = x(u) \in B_1$ for each $u \in W$,*

(ii) *f is continuously differentiable on $B_1 \times W$,*

(iii) *$\| x(u + h) - x(u)\|/\| h \|$ is bounded as $\| h \| \to 0$ for each $u \in W$ and all h such that $u + h \in W$,*

(iv) *A^{-1} exists and is bounded on B_2,*

(v) *$(A - D_1 f(x(u), u; \cdot))^{-1}$ exists and is bounded on B_2 for all $u \in W$, then $x(u)$ has a Fréchet derivative and*

$$Dx(u; h) = (A - D_1 f(x(u), u; \cdot))^{-1} \circ D_2 f(x(u), u; h) \tag{3.3}$$

for all u and $u + h \in W$.

Remarks. (1) Hypothesis (iii) is perhaps the hardest to verify in general, but it can be verified in many practical cases.

(2) Hypothesis (v) naturally follows from (iv) if $D_1 f$ is sufficiently small.

Proof. Consider

$$x(u + h) - x(u) - (A - D_1 f(x(u), u; \cdot))^{-1} \circ D_2 f(x(u), u; h) = \Theta(u, h). \tag{3.4}$$

If $u + h \in W$, then each term on the left of (3.4) is in the domain of A which since $D_1 f(x(u), u; \cdot)$ is bounded, is also the domain of $A - D_1 f(x(u), u; \cdot)$. Applying the latter to (3.4) we get

$$\begin{aligned}
&(A - D_1 f(x(u), u; \cdot))\, \Theta \\
&\quad = Ax(u+h) - Ax(u) - D_1 f(x(u), u; x(u+h) - x(u)) \\
&\qquad - D_2 f(x(u), u; h) \\
&\quad = f(x(u+h), u+h) - f(x(u), u) - D_1 f(x(u), u; x(u+h) - x(u)) \\
&\qquad - D_2 f(x(u), u; h) \\
&\quad = f(x(u) + [x(u+h) - x(u)], u+h) - f(x(u), u) \\
&\qquad - Df(x(u), u; x(u+h) - x(u), h),
\end{aligned}$$

where we have used (2.3). If we denote $(A - D_1 f(x(u), u; \cdot))\, \Theta$ by Φ, then from the definition of the Fréchet derivative for f, we find

$$\lim_{\|x(u+h)-x(u)\|+\|h\|\to 0} \left[\frac{\|\Phi\|}{\|x(u+h) - x(u)\| + \|h\|}\right] = 0.$$

(We have taken the norm in $B_1 \times B_2$ to be $\|\ \|_{B_1} + \|\ \|_{B_3}$.) Note that hypotheses (iii) implies $x(u)$ is continuous, so that the limit makes sense. Further, $\Theta = (A - D_1 f(x(u), u; \cdot))^{-1}\, \Phi$, and hence from hypothesis (v),

$$\|\Theta\| \leqslant K \|\Phi\|. \tag{3.5}$$

Thus

$$\frac{\|\Theta\|}{\|h\|} \leqslant \frac{K\|\Phi\|}{\|h\| + \|x(u+h) - x(u)\|} \left(1 + \frac{\|x(u+h) - x(u)\|}{\|h\|}\right), \tag{3.6}$$

and by hypothesis (iii), the limit of the right hand side of (3.6) exists and is zero. Therefore

$$\lim_{\|h\|\to 0} \frac{\|\Theta\|}{\|h\|} = 0,$$

and it follows from (3.4) that $Dx(u)$ exists and its value is given by (3.3). ∎

COROLLARY 1. *If W is convex, conditions* (iii) *and* (v) *may be replaced by*

(iiia) $$\sup_{u\in W} \|Df(x(u), u; \cdot)\| < \frac{1}{\|A^{-1}\|}.$$

Proof. By the mean value theorem [2, p. 155],

$$\begin{aligned}&\| x(u+h) - x(u)\| \\ &\quad = \| A^{-1}[f(x(u+h), u+h) - f(x(u), u)]\| \\ &\quad \leqslant \| A^{-1} \| \cdot (\| \Delta x \| + \| h \|) \sup_{0 \leqslant \xi \leqslant 1} \| Df(x(u) + \xi \Delta x, u + \xi h; \cdot)\|,\end{aligned}$$

where

$$\Delta x = x(u+h) - x(u).$$

Hence

$$\frac{\| \Delta x \|}{\| h \|} \leqslant \frac{\| A^{-1} \| \sup_{0 \leqslant \xi \leqslant 1} \| Df(x + \xi \Delta x, u + \xi h; \cdot)\|}{1 - \| A^{-1} \| \sup_{0 \leqslant \xi \leqslant 1} \| Df(x + \xi \Delta x, u + \xi h; \cdot)\|}$$

and (iii) follows from (iiia).

Since

$$\| D_1 f \| \leqslant \| Df \|, \qquad \| D_1 f(x(u), u; \cdot)\| < \frac{1}{\| A^{-1} \|},$$

which implies $(A - D_1 f(x(u), u; \cdot)^{-1}$ exists, and is bounded. ∎

We are now in a position to state and prove our principal result. By $\langle \cdot, \cdot \rangle$ we denote the duality between a Banach space and its topological dual.

THEOREM 2. *Let A, f and W be as in Theorem 1. Let F be a continuously differentiable real functional on $B_1 \times W$, and let $U \subset W$. A necessary condition that $\tilde{u} \in U$ be optimal is that for any $y \in B_2{}^*$ (the top dual of B_2) and for any h such that $\tilde{u} + h \in \mathscr{C}_U(\tilde{u}) \cap W$,*

$$\begin{aligned}D_1F(x(\tilde{u}), \tilde{u}; \cdot) \circ Dx(\tilde{u}, h) &+ D_2F(x(\tilde{u}), \tilde{u}; h) \\ &+ \langle y, [A - D_1 f(x(\tilde{u}), \tilde{u}; \cdot)] \circ Dx(\tilde{u}; h)\rangle \\ &- \langle y, D_2 f(x(\tilde{u}), \tilde{u}; h)\rangle \geqslant 0.\end{aligned} \tag{3.7}$$

Further, if there exists a solution $z \in B_2{}^*$ of the conjugate equation

$$(A - D_1 f(x(\tilde{u}), \tilde{u}; \cdot))^* z = - D_1F(x(\tilde{u}), \tilde{u}; \cdot) \tag{3.8}$$

then in order that $\tilde{u}$ be optimal it is necessary that

$$D_2F(x(\tilde{u}), \tilde{u}; h) - \langle z, D_2 f(x(\tilde{u}), \tilde{u}; h)\rangle \geqslant 0. \tag{3.9}$$

Proof. Since $x(u) - A^{-1}f(x(u), u) = \psi \equiv 0$ in B_1, and each part has a derivative with respect to u (recall A^{-1} is bounded),

$$D\psi(u; h) = Dx(u; h) - A^{-1}[D_1 f(x(u), u; \cdot) \circ Dx(u; h) + D_2 f(x(u), u; h)] \equiv 0.$$

Since $Dx(u; h)$ is in the domain of A, for any $y \in B_2^*$,

$$\langle y, ADx(u; h) - D_1 f(x(u), u; \cdot) \circ Dx(u; h) - D_2 f(x(u), u; h) \rangle = 0 \qquad (3.10)$$

for all u and h such that $u, u + h \in W$.

Combining the above with the fact that $\tilde{u} \in U$ minimizes the continuously differentiable functional $F(x(u), u)$ in U, by Lemma 1, for any h such that $\tilde{u} + h \in \mathscr{C}_U(\tilde{u}) \cap \mathscr{W}$

$$\begin{aligned} &D_1F(x(\tilde{u}), \tilde{u}; \cdot) \circ Dx(\tilde{u}; h) + D_2F(x(\tilde{u}), \tilde{u}; h) \\ &\quad + \langle y, ADx(\tilde{u}; h) - D_1 f(x(\tilde{u}), \tilde{u}; \cdot) \circ Dx(\tilde{u}, h) \rangle \\ &\quad - \langle y, D_2 f(x(\tilde{u}), \tilde{u}; h) \rangle \geqslant 0. \end{aligned}$$

If a solution z of (3.8) exists, then

$$\begin{aligned} &\langle z, [A - D_1 f(x(\tilde{u}), \tilde{u}; \cdot)] \circ Dx(\tilde{u}; h) \rangle \\ &\quad = \langle [A - D_1 f(x(\tilde{u}), \tilde{u}; \cdot)]^* z, Dx(\tilde{u}; h) \rangle \\ &\quad = - D_1F(x(\tilde{u}), \tilde{u}; \cdot) \circ Dx(\tilde{u}; h). \end{aligned}$$

Substituting this in (3.7) and replacing y by z, we obtain (3.9). ∎

4. Example 1

We first consider a simple finite dimensional optimization problem. Let A be the operation of rotation in the plane counterclockwise through an angle θ. By $(\cdot, \cdot)$ we denote the inner product in the plane. Let r be a fixed vector, and $\beta > 0$ be given. For our control region we take the closed unit disk with a sector removed:

$$U = \left\{ (\rho, \theta) \,\middle|\, \frac{-\pi}{2} \leqslant \theta \leqslant \frac{5\pi}{4}, \rho \leqslant 1 \right\}.$$

We seek to minimize

$$F(x, u) = (r, x) + \beta(x, x) \qquad (4.1)$$

subject to the condition

$$Ax = u, \qquad u \in U. \qquad (4.2)$$

(This example of Kazarinoff was treated in [8] with a convex control set.) Note that

$$A = A^{*-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

A simple computation shows

$$\begin{aligned} D_1 f(x(\tilde{u}), \tilde{u}; \cdot) &= 0, \\ D_2 f(x(\tilde{u}), \tilde{u}, \cdot) &= I \qquad \text{(the identity operator)}, \\ D_1 F(x(\tilde{u}), \tilde{u}; \cdot) &= r + 2\beta x, \\ D_2 F(x(\tilde{u}), \tilde{u}; h) &= 0. \end{aligned}$$

The conjugate Eq. (3.8) becomes

$$(A - 0)^* z = -(r + 2\beta x)$$

and the necessary condition (3.9) becomes

$$(Ar + 2\beta\tilde{u}, h) \geqslant 0 \qquad \text{for all } h \text{ such that } \tilde{u} + h \in \mathscr{C}_U(\tilde{u}).$$

We choose

$$r = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \beta = \tfrac{1}{2}.$$

If

$$\tilde{u} = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix},$$

the above inequality becomes

$$\left(\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}, u - \tilde{u} \right) \geqslant 0.$$

To further simplify computation, we choose $\Theta = 0$ so that the above reduces to

$$u_1 \geqslant \tilde{u}_1 + (\tilde{u}_1^2 + \tilde{u}_2^2) - (u_1\tilde{u}_1 + u_2\tilde{u}_2). \tag{4.3}$$

By trying various values of u (we do not yet know which $h = u - \tilde{u}$ are admissible), (4.3) forces certain restrictions on $\tilde{u}$. For example, $u = \binom{0}{1}$ yields the condition $\tilde{u}$ lies in the circle

$$(\tilde{u}_1 + \tfrac{1}{2})^2 + (\tilde{u}_2 - \tfrac{1}{2})^2 \leqslant \tfrac{1}{2}.$$

$u = \binom{-1/2}{1/2}$ leads to the condition $\tilde{u}$ must lie in the circle

$$(\tilde{u}_1 + \tfrac{3}{4})^2 + (\tilde{u}_2 - \tfrac{1}{4})^2 \leqslant \tfrac{1}{8}.$$

The intersection of this circle with U is just the point $\binom{-1/2}{1/2}$, which is indeed the optimal control. It should be noted that there is one other local minimum, namely, the origin, and the choice of a direction h that is admissible for both points $\binom{-1/2}{1/2}$ and $\binom{0}{0}$ leads to a circle containing both points. If a direction is chosen that is not admissible for either of these $\tilde{u}$, then the necessary conditions lead to a contradiction such as that $\tilde{u} \notin U$.

4. Example 2

We consider a simple Dirichlet Problem for the Laplacian. Let Ω be the unit disk in the plane, and let A be the Friedrichs self adjoint extension of the Laplacian with zero boundary conditions. Let U be the closed unit ball in $L^2(\Omega)$, r be a fixed vector in $L^2(\Omega)$, and $\beta > 0$. We consider the problem: To minimize

$$F(x, u) = (r, x) + \beta(x, x) \tag{4.4}$$

subject to the condition

$$Ax = u, \qquad u \in U, \tag{4.5}$$

$(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$. We take $B_1 = B_2 = B_3 = L^2(\Omega)$. It is well known that for each u there exists a unique solution $x(u) \in \operatorname{dom}(A)$, and, by the weak lower semicontinuity of F and the convexity of U, the existence of an optimal control is guaranteed (cf. Lions [6]).

Because A^{-1} is bounded and $D_1 f(x(u), u; \cdot) = 0$, all of the conditions of Theorems 1 and 2 are easily seen to be satisfied. The conjugate Eq. (3.8) becomes

$$A^* z = - D_1 F(x(\tilde{u}), \tilde{u}, \cdot) = - (r + 2\beta\tilde{x}), \qquad \tilde{x} = x(\tilde{u}). \tag{4.6}$$

(We identify L^2 with its dual.)

The necessary condition (3.9) becomes

$$(A^{*-1}[r + 2\beta\tilde{x}], u - \tilde{u}) \geqslant 0 \tag{4.7}$$

for all admissible $h = u - \tilde{u}$, $\tilde{u} + h \in \mathscr{C}_U(\tilde{u})$.

The complete solution to the problem involves the simultaneous solution of (4.5)–(4.7), which, because of the simplicity of (4.5), would be feasible.

The necessary conditions alone, however, will in some instances give useful information, such as whether the optimal control in bang-bang ($\| \tilde{u} \| = 1$).

If $\tilde{u} \in \operatorname{int}(U)$, then all directions are admissible, and it follows from (4.7) that

$$A^{-1}((r/2\beta) + \tilde{x}) = 0.$$

Since A^{-1} has kernel zero, this would imply

$$\tilde{x} = -r/2\beta,$$
$$\tilde{u} = A\tilde{x} = -Ar/2\beta.$$

But this could not be satisfied unless r were in the domain of A (it was originally just assumed in $L^2(\Omega)$). As is well known, $\mathscr{D}(A) \subset \mathring{H}^1(\Omega)$. Thus if r is not sufficiently smooth, $\tilde{u}$ must be on the boundary of the unit ball, and the process is bang-bang. Moreover, even if $r \in \mathscr{D}(A)$, it follows that

$$\| \tilde{u} \| = \left\| \frac{Ar}{2\beta} \right\|, \qquad \text{and} \qquad \| \tilde{u} \| < 1 \qquad \text{only if} \qquad \| Ar \| < 2\beta.$$

We thus conclude that the process will be bang-bang if either

(i) r is not in $\mathscr{D}(A)$,

(ii) if $r \in \mathscr{D}(A)$, and $\| Ar \| \geqslant 2\beta$.

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